Introduction to Probability

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PRIMES Circle

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Introduction of Topics

Probability

Probability is how likely an event is going to happen. There are many different ways to determine the probability of something, such as combinations, conditional probability, Bayes' Theorem, etc.

Combinations

Where $n! = n * (n-1) * (n-2) * \ldots * 3 * 2 * 1$:

$$ {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} $$

Introduction of Topics

Conditional Probability

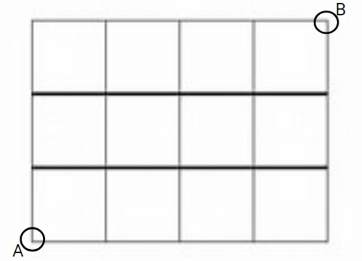
The probability of an event occurring, given that the event has already occurred. For example, $P(A|B) =$ The probability of event A happening given that event B has already happened.

Complementary Counting

Involves counting what you don't want, and subtracting that from the total number of possibilities.

The Grid Problem

Consider the grid of points shown at the top of the next column. Suppose that, starting at the point labeled A, you can go one step up or one step to the right at each move. This procedure is continued until the point labeled B is reached. How many different paths from A to B are possible?

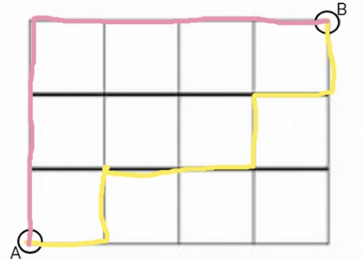


The Grid Problem Solution

There are 3 steps up and 4 steps to the right, so the total number of moves needed is always 7. We have 7 moves, so we need to choose 3 moves to go up (alternatively, you can choose to move 4 spaces to the right. This is the same as ${}_7C_4$, which is equivalent to 35). We can solve this problem using the combination formula.

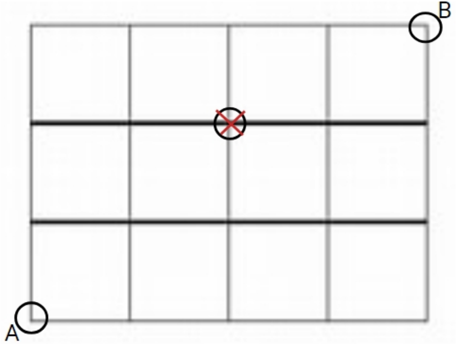
$$ {}_7C_4 = {}_7C_3 = 35 $$

There are 35 different paths from point A to point B from the grid.



The Grid Problem With Obstacle

From the same grid as before, how many different paths are there from A to B that do not go through the point crossed in the following lattice?



The Grid Problem With Obstacle Solution

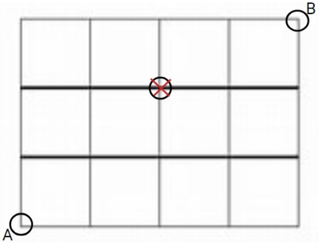
(# Paths from A to crossed point) $*$ (# Paths from crossed point to B):

$$_4C_2 *_3 C_2 = 12$$

$$4!/2!(4-2)! * 3!/2!(3-2)! = 12$$

$$35 - 12 = 23$$

Therefore, there are 23 different paths from point A to B that avoid the crossed point.



Conditional Probability

Conditional probabilities are used to compute the desired probabilities more easily when no partial information is available.

Definition:

$P(E|F) = \frac{P(EF)}{P(F)}$

Example:

Say you want to find the probability of rolling a 2 on a dice given you rolled only even numbers. E would represent rolling a 2, and F would be rolling even numbers only.

$$\text{P(EF)} = 1/6, \text{ and P(F)} = 1/2. \text{ Thus, } \frac{1/6}{1/2} = 1/3.$$

Example 1: Problem using Conditional Probability

A student is planning to take her three actuarial examinations in the summer.

- She will take the first actuarial exam in June.
- If she passes that exam, then she will take the second exam in July.
- If she also passes that one, then she will take the third exam in September.

If she fails an exam, then she is not allowed to take any others.

- The probability that she passes the first exam is .9.
- If she passes the first exam, then the conditional probability that she passes the second one is .8.
- If she passes both the first and second exams, then the conditional probability that she passes the third exam is .7.

Example 1: Problem using Conditional Probability

Part a

What is the probability that she passes all three exams?

We know that:

- She needs to pass the previous exam in order to take the next.
- P(Pass 1st exam) = 0.9, P(Pass 2nd exam) = 0.8, P(Pass 3rd exam) = 0.7

Thus,

$$P(\text{Passes all three exams}) = 0.9 * 0.8 * 0.7 = 50.4 \text{ percent.}$$

Example 1: Problem using Conditional Probability

Part b

Given that she did not pass all three exams, what is the conditional probability that she failed the second exam?

We will use the formula to solve for conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

We define $A =$ Failing the 2nd exam, $B =$ Not passing all three exams.

$$P(A|B) = \frac{P(\text{Fail second exam and not pass all 3 exams})}{P(\text{Not pass all 3 exams})}$$

$$P(A|B) = \frac{0.9*0.2}{1-(0.9*0.8*0.7)} = 36.29 \text{ percent}$$

Deriving Bayes' Theorem

Conditional Probability Formula:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

We can also say that:

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

We know that these two equal each other due to commutative property of union and intersection:

$$P(A \cap B) = P(B \cap A)$$

Using basic algebra:

$$P(A|B) * P(B) = P(B|A) * P(A)$$

Results in:

$$P(A|B) = \frac{P(B|A) * P(A)}{P(B)}$$

Which is Bayes' Theorem.

Example 2: Problem using Bayes' Theorem

A total of 46 percent of the voters in a certain city classify themselves as Independents, whereas 30 percent classify themselves as Liberals and 24 percent say that they are Conservatives.

In a recent local election, 35 percent of the Independents, 62 percent of the Liberals, and 58 percent of the Conservatives voted. If a voter is chosen at random given that this person voted in the local election, what is the probability that they are:

a) an Independent?
b) a Liberal?
c) a Conservative?
d) What percent of voters participated in the local election?

Example 2: Problem using Bayes' Theorem

Given information:

$$P(\text{Independent}) = 0.46$$
$$P(\text{Liberal}) = 0.30$$
$$P(\text{Conservative}) = 0.24$$

We define $V =$ voted:

$$P(V|I) = 0.35$$
$$P(V|L) = 0.62$$
$$P(V|C) = 0.58$$

Example 2: Problem using Bayes' Theorem

Part d)

What percent of voters participated in the local election?

We solve for part d first because in order to solve the other parts, we need to know the total number of people who voted in the election.

We need to use a formula for independence which states that

$$P(E) = P(E|F)P(F) + P(E|G)P(G) + P(E|H)P(H)$$

In this problem:

$$P(V) = P(V|I)P(I) + P(V|L)P(L) + P(V|C)P(C)$$

$$P(V) = 0.161 + 0.186 + 0.1392$$

$$P(V) = 0.4862$$

Example 2: Problem using Bayes' Theorem

- Given that a person voted in the local election, what is the probability that they are an Independent?

$$P(I|V) = (P(I) * P(V|I))/P(V)$$
$$P(I|V) = (0.46 * 0.35)/0.4862$$
$$P(I|V) = 0.331$$

- What is the probability that they are a Liberal?

$$P(L|V) = (P(L) * P(V|L))/P(V)$$
$$P(L|V) = (0.30 * 0.62)/0.4862$$
$$P(L|V) = 0.383$$

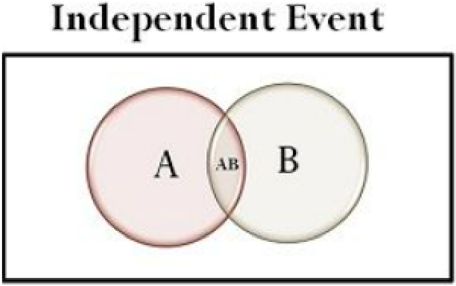
- What is the probability that they are a Conservative?

$$P(C|V) = (P(C) * P(V|C))/P(V)$$
$$P(C|V) = (0.24 * 0.58)/0.4862$$
$$P(C|V) = 0.286$$

Independent Events

Definition:

Two events E and F are said to be independent if $P(EF) = P(E)P(F)$ holds. Two events E and F that are not independent are said to be dependent.



Example 3: Two events that are not independent

Suppose that we toss 2 fair dice. Let E denote the event that the sum of the dice is 6 and F denote the event that the first die equals 4. Then

$$P(EF) = (4, 2) = (1/6)(1/6) = 1/36$$

whereas

$$P(E)P(F) = (5/36)(1/6) = 5/216$$

Hence, E and F are not independent because $P(EF) \neq P(E)P(F)$.

Example 4: Using Independence in a real-life example

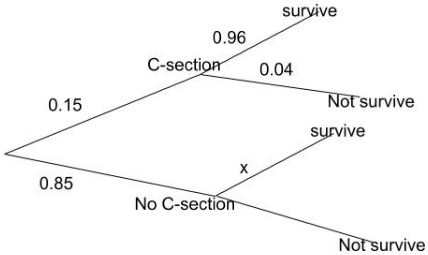
Ninety-eight percent of all babies survive surgery. However, 15 percent of all births involved Cesarean (C) sections, and when a C section is performed, the baby survives 96 percent of the time.

If a randomly chosen pregnant woman does not have a C section, what is the probability that her baby survives?

Example 4: Using Independence in a real-life example

Given information:

- 98% of all babies survive delivery.
- 15% of all births involved C sections.
- When a C section is performed, the baby survives 96% of the time.



$$0.98 = (0.15 * 0.96) + (x * 0.85)$$
$$x = 0.9835 = 98.35\%$$

Acknowledgements

- Ross Sheldon, *A First Course in Probability*, New York, 1984.

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